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Eigenvector Error Bounds and Their Applications to Structural Modification

Yitshak M. Ram*
University of Adelaide, Adelaide 5001, Australia
and
Simon G. Braun†
Technion—Israel Institute of Technology, Haifa 32000, Israel

We have shown previously that some natural frequencies and mode shapes of a modified vibrating structure may be approximated from the incremental mass and stiffness matrices, and from an incomplete set of measured natural frequencies and mode shapes for the unmodified system. In addition, error bounds were obtained for the natural frequencies of the modified structure. We bound here the error in the approximated mode shapes.

I. Introduction

A FREQUENTLY encountered engineering problem is to predict the effect of structural modifications on the modal parameters of a structure. Apparently, this problem can be solved analytically by establishing a discrete finite element model for the modified structure. From the theoretical standpoint, the process of developing analytical discrete models (e.g., a finite element model) associated with continuous system is well developed and known. However, to actually implement these methods in practice, it is required to determine for each element of the discrete model some physical properties, e.g., density, Young's modulus of elasticity, and cross-sectional moments of inertia. In most practical applications these values are not readily available, and, consequently, intuitive values are used instead. Hence, the resulting analytical models are subject to considerable uncertainty. Quite clearly, the modal parameters that may be derived from these models will suffer from large inaccuracy as well. The difficulty of evaluating an analytical model for the entire modified structure may be avoided if the modal parameters of the modified structure are approximated from 1) experimental modal testing results of the unmodified structure, and 2) an analytical model for the incremental part only (of the modified structure).

Attention will now be focused on the feasibility of this approach. For the purpose of the analysis it is assumed that the structure may be accurately represented by a large, mth order, discrete model. Therefore, to evaluate the required model experimentally, it is necessary to measure the dynamic response of the structure at m different points. These measurements may be obtained by using the experimental modal analysis techniques. However, as measurement technologies (at least, in the present state of the art) are limited to the extraction of a relatively small number $n (n \ll m)$ of natural frequencies and mode shapes, it is generally acceptable (e.g., Berman[1]) that "a valid physical model cannot be identified from test data alone." Although progress in the problem of evaluating the physical parameters of simple models, e.g., mass-spring systems and vibrating rods or beams, has been obtained recently,[2-4] the modal parameters of a general, complex, modified structure cannot as yet be determined in a direct manner from experimental modal data of the unmodified structure and the analytical model for the incremental portion. Nevertheless, we have developed in Ref. 5 a method for approximating the modal parameters of a structure based on such data and have shown that the approximation is optimal in a Rayleigh-Ritz sense.

The associated *inverse* problem, in which the incremental matrices are derived to meet prescribed modal data, is posed and also partially solved in Ref. 6. Naturally, it is of importance to investigate the question "How good are these approximations?" Clearly, several error mechanisms contribute to the discrepancy between the modal parameters of the real structure and their analytical approximation. First, the real structure is a nonlinear continuous system, whereas the model is linear and discrete. Second, there are uncertainties regarding the physical properties and boundary conditions of the vibrating structure. Third, the approximation is based on incomplete modal data for the unmodified structure, e.g., only n natural frequencies and mode shapes are assumed to be known. We have named this last effect, the modal truncation error. The analytical approximation is based mainly on data from experiments, and we assume that the mechanisms of error resulting from nonlinearity, discretization, and uncertainties in the physical parameters are negligible with respect to the dominant error caused by the modal truncation. Under such an assumption the possible errors of the natural frequencies of the modified structure have been bounded, using various methods, in a series of papers.[7-10] The estimation of natural frequences and their range of variation is a main objective in the design and analysis of vibratory systems. For some applications it is equally important to be able to predict mode shapes and error bounds for the modified structure. This information is essential for the analysis of the dynamic stresses and strains. Bounding the nodal points and the extreme values of the fundamental mode shapes is also of great interest for design purposes.

The objective of this paper is to derive bounds on the error between the n mode shapes of the structure and their Rayleigh-Ritz approximations. Davis and Kahan[11] have derived bounds on the rotational angle of eigenvectors caused by perturbations, and a portion of their theory is employed here. However, to evaluate these bounds it is required to determine a certain spectral gap and a residual matrix norm. Since it is assumed that the stiffness matrix of the structure is unknown, the residual matrix cannot be found from the formula given by Davis and Kahan. The main contribution here will be in developing an alternative formula for the residual matrix and in applying previous results[7-9] to bound the spectral gap. The authors have, in fact, used a similar approach to bound eigenvectors in Ref. 7. However, the results obtained there were restricted to the situation in which the structural modification affects the stiffness matrix but leaves the mass matrix un-

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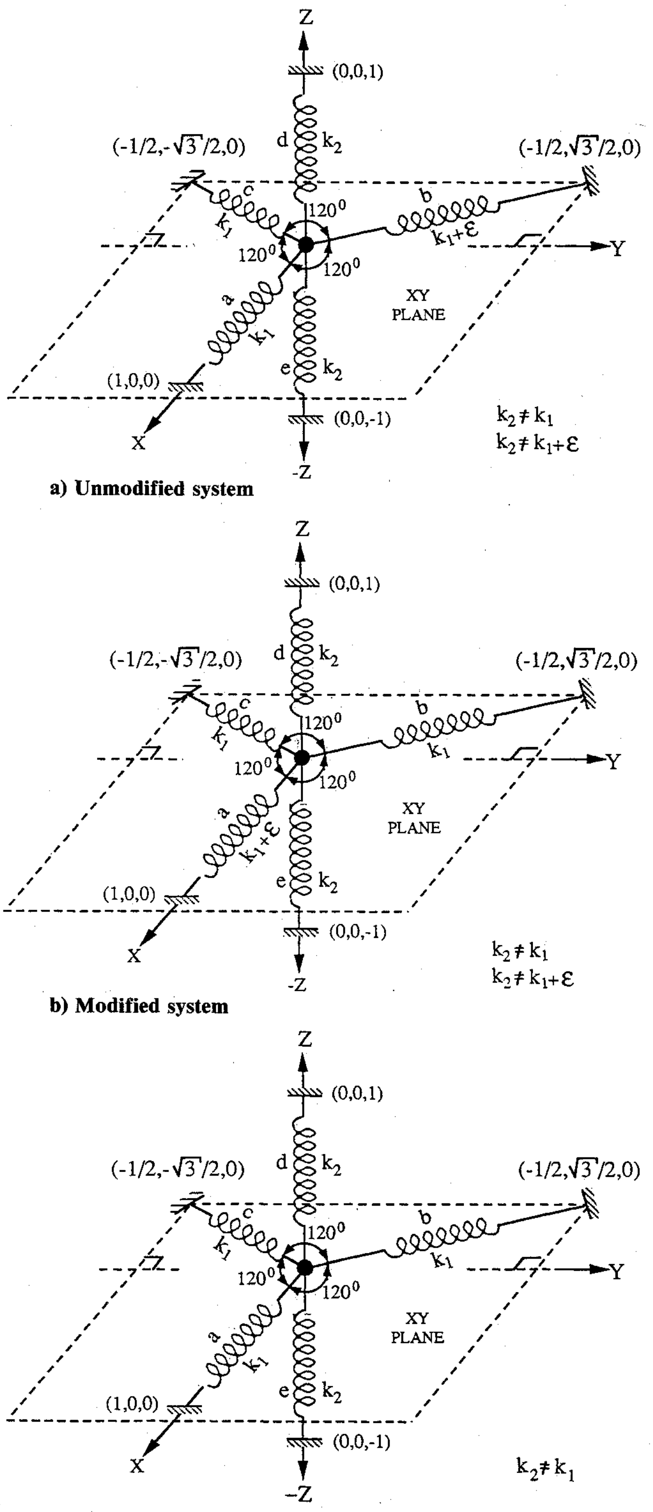
*Lecturer, Mechanical Engineering Department.
†Professor, Faculty of Mechanical Engineering.

changed. Here the developments are not restricted by this limitation.

The contents of the paper appear in the following order. The relevant background is covered in Sec. II, the problem formulation is presented in Sec. III, the main results are derived in Sec. IV, and an example is given in Sec. V.

II. Mathematical Background

It is well known that an eigenvector can be normalized arbitrarily. If x is an eigenvector of $(A - \lambda B)x = 0$, then αx is also an eigenvector of the same eigenvalue problem, where α

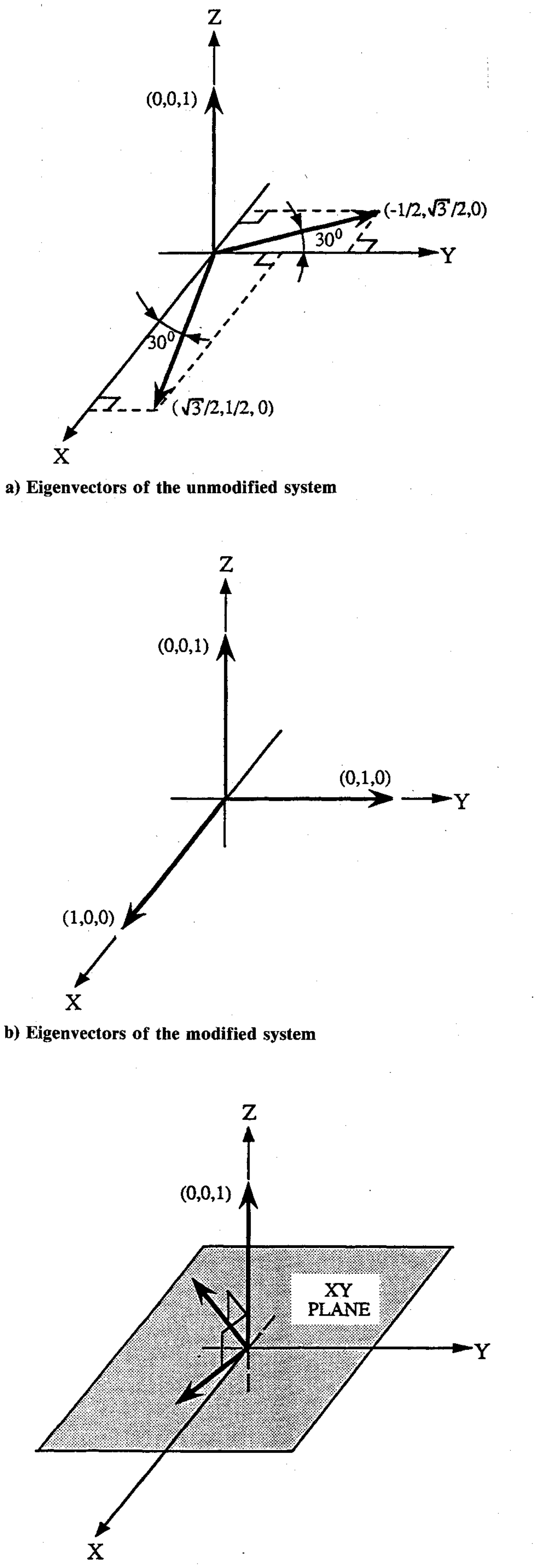


a) Unmodified system

b) Modified system

c) System with two identical natural frequencies

Fig. 1 Vibratory systems with two poorly separated natural frequencies.



a) Eigenvectors of the unmodified system

b) Eigenvectors of the modified system

c) Eigenvectors of the system with two identical natural frequencies

Fig. 2 The eigenvectors corresponding to the systems shown in Fig. 1.

is an arbitrary nonzero scalar. Hence, to bound the error between an eigenvector and its approximation, it is enough to determine the maximum angle between these two vectors. Consider now the associated modified problem $(A + \Delta A)\hat{x} = \hat{\lambda}\ (B + \Delta B)\hat{x}$. For infinitesimal perturbations, where $\Delta M \rightarrow 0$ and $\Delta K \rightarrow 0$, we necessarily have $\hat{\lambda} \rightarrow \lambda$. However, in the case where λ is not a simple eigenvalue (i.e., a multiple eigenvalue) the eigenvector x of $(A - \lambda B)x = 0$ is not uniquely determined (up to a normalized constant). Hence, it does not necessarily follow that $\hat{x} \rightarrow \beta x$ (where β is a constant), unless λ is a simple eigenvalue. It may be deduced that when λ is a well-separated eigenvalue, its corresponding eigenvector x is insensitive to perturbations in A and B, in the sense that infinitely small modifications in the elements of these matrices will lead to an infinitely small change in the orientation of x.

However, when λ is not well separated from the other eigenvalues, its corresponding eigenvector x may be sensitive to perturbations, as demonstrated in the following example. Consider the three degree-of-freedom system shown in Fig. 1a. The mass normalized eigenvectors of this system are $(-1/2, \sqrt{3}/2, 0)^T$, $(\sqrt{3}/2, 1/2, 0)^T$, and $(0, 0, 1)^T$, as shown in Fig. 2a. Suppose that by reducing the spring's constant b to k_1 and increasing a to $k_1 + \epsilon$ the modified system is determined (Fig. 1b). Then the vectors $(1, 0, 0)^T$, $(0, 1, 0)^T$, and $(0, 0, 1)^T$, shown in Fig. 2b, are the eigenvectors of the modified system. Note, that in this case, the elements of the stiffness matrix have been changed by quantities of order ϵ, and the mass matrix remains unchanged. Hence, for any ϵ, the eigenvectors $(-1/2, \sqrt{3}/2, 0)^T$ and $(\sqrt{3}/2, 1/2, 0)^T$ rotate around $(0, 0, 1)^T$ by an angle of $\pi/6$. Consequently, when $\epsilon \rightarrow 0$, an infinitely small modification causes a finite change in the orientation of the eigenvectors.

It is noted that when $\epsilon = 0$ (Fig. 1c), the system has a twofold eigenvalue. Hence, each vector which lies in the x-y plane (see Fig. 2c) is an eigenvector of the same problem. The effect of the modification, no matter how small, is that the repeated eigenvalues separate into two distinct eigenvalues; and the eigenvectors are determined uniquely. However, it will be shown in Sec. IV that the sensitive eigenvectors, in the case of a poorly separated eigenvalue, span a subspace which is insensitive to perturbations in A and B. Hence, for small perturbations the new eigenvectors lie near the subspace that is spanned by the eigenvectors corresponding to the repeated eigenvalue. Consequently, when some of the eigenvalues are clustered within a small interval, it is required to bound the "distance" between the subspace that is spanned by the perturbed eigenvectors corresponding to the poorly separated eigenvalues and its Rayleigh Ritz approximation.

The preceding discussion leads the investigation toward the relationship between two subspaces. Suppose U and V are two $m \times q$ $(m > q)$ orthonormal matrices, and let the orthonormal columns of U and V span the subspaces $\mathcal{U}$ and $\mathcal{V}$, respectively. Then the principal angles $\theta_1, \ldots, \theta_q$ between $\mathcal{U}$ and $\mathcal{V}$ are defined by

$$\theta_i \equiv \cos^{-1}\sigma_i\,(U^TV), \qquad \theta_i \epsilon [0, \pi/2], \qquad i = 1, \ldots, q \quad (1)$$

where $\sigma_i\,(U^TV)$ is the ith largest singular value of U^TV. The greatest principal angle θ_q is called the angle between the subspaces. When $q = 1$, Eq. (1) reduces to $\theta_1 = \cos^{-1} U^TV$, which is the classical formula for the angle between two unit vectors U and V.

The following example permits a geometrical interpretation of these definitions. Let

$$U = \begin{bmatrix} \sqrt{2}/2 & \sqrt{2}/2 \\ \sqrt{2}/2 & -\sqrt{2}/2 \\ 0 & 0 \end{bmatrix} \qquad V = \begin{bmatrix} 0 & \sqrt{2}/2 \\ 1 & 0 \\ 0 & \sqrt{2}/2 \end{bmatrix}$$

Then

$$U^TV = \begin{bmatrix} \sqrt{2}/2 & 1/2 \\ -\sqrt{2}/2 & 1/2 \end{bmatrix}$$

which has singular values $\sigma_1\,(U^TV) = 1$ and $\sigma_2(U^TV) = \sqrt{2}/2$. It follows from Eq. (1) that the principal angles between $\mathcal{U}$ and $\mathcal{V}$ are $\theta_1 = 0$ and $\theta_2 = \pi/4$. The angle between the subspaces is thus $\pi/4$. Figure 3 illustrates that the angle between $\mathcal{U}$ and $\mathcal{V}$ is indeed $\pi/4$. Hence, the angle between two subspaces as defined by Eq. (1) equates to our geometrical interpretation in three-dimensional space.

Consider, again, the mass-spring system of Fig. 1. Let U be the subspace spanned by the two sensitive eigenvectors of the original system, and let V be the associated subspace corresponding to the modified system. Then,

$$U = \begin{bmatrix} -1/2 & \sqrt{3}/2 \\ \sqrt{3}/2 & 1/2 \\ 0 & 0 \end{bmatrix} \qquad V = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

hence

$$U^TV = \begin{bmatrix} -1/2 & \sqrt{3}/2 \\ \sqrt{3}/2 & 1/2 \end{bmatrix}$$

and $\sigma_1(U^TV) = \sigma_2(U^TV) = 1$. It follows, therefore, from Eq. (1) that $\theta_1 = \theta_2 = 0$, i.e., $\mathcal{U}$ and $\mathcal{V}$ are actually the same subspace. Hence, although each eigenvector of U is highly sensitive to perturbation, the subspace $\mathcal{U}$ spanned by these eigenvectors is absolutely robust with respect to the described modification.

III. Problem Formulation

Consider the m degree-of-freedom system that is characterized by the symmetric definite eigenvalue problem

$$K\Phi = M\Phi\Lambda; \qquad \Phi^TM\Phi = I_m \qquad (2)$$

where M and K are the mass and stiffness matrices of the structure, respectively; $\Lambda = \text{diag}\{\lambda_i(K,M);\ i = 1, \ldots, m\}$ and $\Phi \,\epsilon\, \mathbb{R}^{m \times m}$ is the mass normalized modal matrix. Let $\Phi \equiv [\phi_1 \mid \phi_2 \mid \ldots \mid \phi_m]$ be the column partitioning of Φ. Then ϕ_i is the ith M-orthonormal eigenvector of Eq. (2) corresponding to $\lambda_i\,(K, M)$. Suppose that n eigenvalues λ_i and their corresponding eigenvectors ϕ_i are available from measurements, and let $\Phi_1 \equiv [\phi_1 \mid \ldots \mid \phi_n]$ and $\Lambda_1 \equiv \text{diag}\ \{\lambda_i;\ i = 1, \ldots, n\}$. For simplicity, it is further assumed that a good approximation for the mass matrix M can be found using the finite element method, whereas the stiffness matrix K is unknown. Let ΔM and ΔK be the incremental mass and stiffness matrices resulting from the modification. Then, the eigenvalue problem of the modified structure is

$$(K + \Delta K)\hat{\Phi} = (M + \Delta M)\hat{\Phi}\hat{\Lambda}; \qquad \hat{\Phi}^T(M + \Delta M)\hat{\Phi} = I_m$$
$$\hat{\Lambda} = \text{diag}\ \{\hat{\lambda}_i;\ i = 1, \ldots, n\} \qquad (3)$$

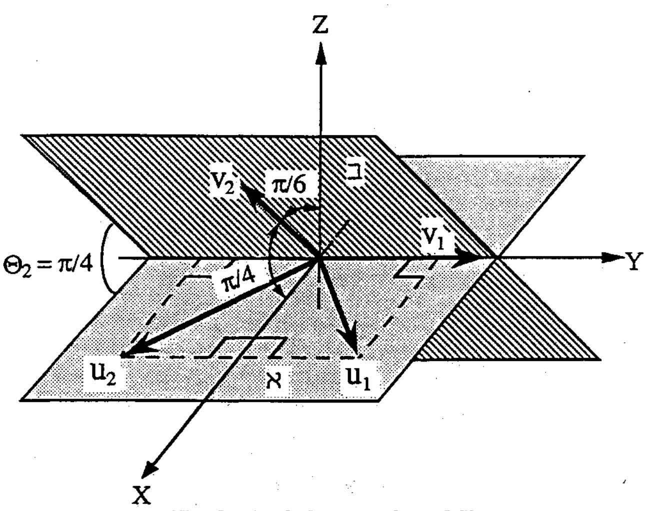


Fig. 3 Angle between $\mathcal{U}$ and $\mathcal{V}$.

or, equivalently,

$$AW = W\Omega \tag{4}$$

where $A \equiv (M + \Delta M)^{-1/2}(K + \Delta K)(M + \Delta M)^{-1/2}$, $W = (M + \Delta M)^{1/2}\hat{\Phi}$, and $\Omega \equiv \hat{\Lambda}$. Let F and $\tilde{\Omega}$ be the nontrivial solution of

$$(\Lambda_1 + \Phi_1^T \Delta K \Phi_1)F = (I_n + \Phi_1^T \Delta M \Phi_1)F\tilde{\Omega}$$

$$F^T(I_n + \Phi_1^T \Delta M \Phi_1)F = I_n$$

$$\tilde{\Omega} \equiv \text{diag}\ \{\tilde{\lambda}_i,\ i = 1, \ldots, n\} \tag{5}$$

and, by definition, introduce

$$\tilde{W} \equiv (M + \Delta M)^{1/2}\Phi_1 F \tag{6}$$

Let $W \equiv [w_1, \ldots, w_m]$, $\tilde{W} \equiv [\tilde{w}_1, \ldots, \tilde{w}_n]$, and $F \equiv [f_1, \ldots, f_n]$ be the column partitioning of W, $\tilde{W}$, and F, respectively. We also define $W_{pq} \equiv [w_p, \ldots, w_{p+q-1}]$, $\tilde{W}_{pq} \equiv [\tilde{w}_p, \ldots, \tilde{w}_{p+q-1}]$, $F_{pq} \equiv [f_p, \ldots, f_{p+q-1}]$, Ω_{pq} diag $\{\hat{\lambda}_i;\ i = p, \ldots, p + q - 1\}$, and $\tilde{\Omega}_{pq} \equiv \text{diag}\ \{\tilde{\lambda}_i;\ i = p, \ldots, p + q - 1\}$. It has been shown in Ref. 5 that $\tilde{W}$ and $\tilde{\Omega}$ are the Rayleigh-Ritz approximations for W_{1n} and Ω_{1n} from SPAN (Φ_1), respectively. It also follows from the preceding definitions that

$$\tilde{W}_{pq} \equiv (M + \Delta M)^{1/2}\Phi_1 F_{pq} \tag{7}$$

Our problem can now be expressed in the following form. Problem definition: Suppose M, Φ_1, Λ_1, ΔM, and ΔK are given, and that the stiffness matrix K is unknown. Let $\mathcal{W}$ and $\tilde{\mathcal{W}}$ be the q-dimensional subspaces that are spanned by W_{pq} and its approximation $\tilde{W}_{pq}$ respectively. The objective of the paper is to bound the angle between $\mathcal{W}$ and $\tilde{\mathcal{W}}$.

The columns of the matrix product $\Phi_1 F$ are the Ritz vectors of the modified system. The matrices W_{pq} and $\tilde{W}_{pq}$ are, therefore, by their definitions [see Eqs. (4) and (7)] mass normalized modal matrices of the modified system and its Rayleigh Ritz approximation, respectively. Thus, we are seeking a bound on the approximation error of the mode shapes of the modified system. Note, that the given data permit the evaluation of F, $\tilde{W}$, and $\tilde{\Omega}$ explicitly by use of Eqs. (5) and (6).

IV. Analysis

The principal results of this section are based on the Davis and Kahan sin θ Theorem.[11] According to this theorem the bound is equal to the ratio of a certain residual matrix norm and spectral gap. Let us denote the spectral gap between the cluster of q Ritz values, starting from the pth eigenvalue, and the complementary spectrum of their corresponding eigenvalues by

$$g_{pq} = \max\ [0, \alpha_{pq}] \tag{8}$$

where

$$\alpha_{pq} = \begin{cases} \hat{\lambda}_{q+1} - \tilde{\lambda}_q; & p = 1 \\ \min\ [\tilde{\lambda}_p - \hat{\lambda}_{p-1},\ \hat{\lambda}_{p+q}\tilde{\lambda}_{p+q-1}]; & p < 1 \end{cases} \tag{9}$$

A geometrical illustration for q_{pq} is shown in Fig. 4a. The residual matrix associated with A and its approximated modal matrix $\tilde{W}_{pq}$ is defined as

$$R[A, \tilde{W}_{pq}] \equiv A\tilde{W}_{pq} - \tilde{W}_{pq}\tilde{W}_{pq}^T A \tilde{W}_{pq} \tag{10}$$

The sin θ theorem states that if $\|R[A, \tilde{W}_{pq}]\|_2 < g_{pq}$ then the angle between the subspace spanned by $\tilde{W}_{pq}$ and the subspace, which is spanned by the corresponding eigenvectors of A, must be smaller than $\sin^{-1}(\|R[A, \tilde{W}_{pq}]\|_2 / g_{pq})$. It is impossible to directly apply the theorem in this analysis for the following two reasons:

1) The eigenvalues of the modified system $(\hat{\lambda}_i;\ i = 1, 2, \ldots)$ are unknown. Therefore, g_{pq} cannot be evaluated from its definition [given by Eqs. (8) and (9)].

2) Since A is unknown, $\|R[A, \tilde{W}_{pq}]\|_2$ cannot be calculated using Eq. (10).

The first difficulty can be circumvented by bounding g_{pq} using the previously determined results in Refs. 8 and 9, as follows. The authors have shown in these papers how to find upper and lower bounds for the eigenvalues of the modified system. So, let λ_i^U and λ_i^L be the upper and lower bounds of the ith smallest eigenvalue of the modified system. Define a new scalar $\hat{g}_{pq}$ by

$$\hat{g}_{pq} = \max\ [0, \beta_{pq}] \tag{11}$$

where

$$\beta_{pq} = \begin{cases} \lambda_{q+1}^L - \tilde{\lambda}_q; & p = 1 \\ \min\ [\tilde{\lambda}_p - \lambda_{p-1}^U,\ \lambda_{p+q}^L\tilde{\lambda}_{p+q-1}]; & p < 1 \end{cases} \tag{12}$$

The spectral gap $\hat{g}_{pq}$ is illustrated geometrically by Fig. 4b. Since $\lambda_i^L \leq \hat{\lambda}_i \leq \lambda_i^U$ it follows that $\hat{g}_{pq} \leq g_{pq}$, and, therefore,

$$\sin^{-1}\frac{\|R[A, \tilde{W}_{pq}]\|_2}{g_{pq}} \leq \sin^{-1}\frac{\|R[A, \tilde{W}_{pq}]\|_2}{\hat{g}_{pq}}$$

The following proposition is now presented to circumvent the second difficulty in evaluating $\|R[A, \tilde{W}_{pq}]\|_2$.

Proposition:

$$\begin{aligned} R[A, \tilde{W}_{pq}] &= [(M + \Delta M)^{-1/2} - \tilde{W}_{pq}\tilde{W}_{pq}^T(M + \Delta M)^{-1/2}]\, M\Phi_1\Lambda_1 F_{pq} \\ &+ (M + \Delta M)^{-1/2}\,\Delta K\,(M + \Delta M)^{-1/2}\tilde{W}_{pq} \\ &- \tilde{W}_{pq}\tilde{W}_{pq}^T(M + \Delta M)^{-1/2}\,\Delta K\,(M + \Delta M)^{-1/2}\,\tilde{W}_{pq} \end{aligned}$$

Proof: Let $P = (M + \Delta M)^{-1/2}K(M + \Delta M)^{-1/2}$ and $\Delta P = (M + \Delta M)^{-1/2}\Delta K(M + \Delta M)^{-1/2}$. Then, by the definition of the residual matrix we have

$$R[A, \tilde{W}_{pq}] = R[P, \tilde{W}_{pq}] + R[\Delta P, \tilde{W}_{pq}] \tag{13}$$

Since

$$\begin{aligned} R[P, \tilde{W}_{pq}] &\equiv (M + \Delta M)^{-1/2}K(M + \Delta M)^{-1/2}\tilde{W}_{pq} \\ &- \tilde{W}_{pq}\tilde{W}_{pq}^T(M + \Delta M)^{-1/2}K(M + \Delta M)^{-1/2}\tilde{W}_{pq} \end{aligned} \tag{14}$$

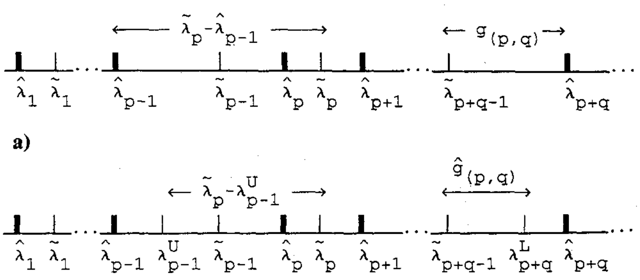


Fig. 4 Gaps in the spectrum: a) g_{pq} and b) $\hat{g}_{pq}$.

it follows from Eq. (7) that

$$R[P, \tilde{W}_{pq}]$$
$$= (M + \Delta M)^{-1/2} K \Phi_1 F_{pq}$$
$$- \tilde{W}_{pq} \tilde{W}_{pq}^T (M + \Delta M)^{-1/2} K \Phi_1 F_{pq}$$
$$= [(M + \Delta M)^{-1/2} - \tilde{W}_{pq} \tilde{W}_{pq}^T (M + \Delta M)^{-1/2}] K \Phi_1 F_{pq} \tag{15}$$

By using

$$K\Phi_1 = M\Phi_1\Lambda_1 \tag{16}$$

we obtain

$$R[P, \tilde{W}_{pq}]$$
$$= [(M + \Delta M)^{-1/2} - \tilde{W}_{pq}\tilde{W}_{pq}^T(M + \Delta M)^{-1/2}] M\Phi_1\Lambda_1 F_{pq} \tag{17}$$

and by definition

$$R[\Delta P, \tilde{W}_{pq}] \equiv (M + \Delta M)^{-1/2} \Delta K (M + \Delta M)^{-1/2} \tilde{W}_{pq}$$
$$- \tilde{W}_{pq}\tilde{W}_{pq}^T (M + \Delta M)^{-1/2} \Delta K (M + \Delta M)^{-1/2} \tilde{W}_{pq} \tag{18}$$

The proof is completed by substituting Eqs. (17) and (18) in Eq. (13).

Note that the proposition enables us to evaluate $R[A, \tilde{W}_{pq}]$ from the given data. It is now possible to bound the required angle by using the following result.

Main theorem: Let θ be the angle between the q-dimensional subspaces spanned by the columns of W_{pq} and $\tilde{W}_{pq}$. If $R[A, \tilde{W}_{pq}] < \hat{g}_{pq}$, then

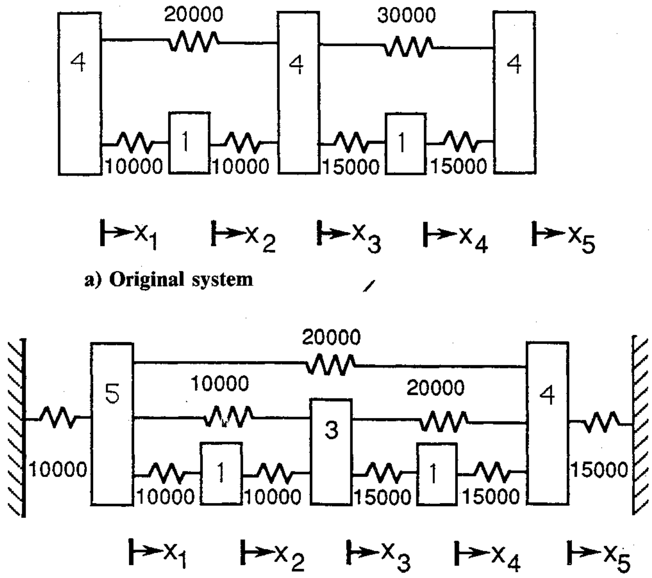
$$\theta \leq \sin^{-1} \frac{\| R[A, \tilde{W}_{pq}] \|_2}{\hat{g}_{pq}} \tag{19}$$

Proof: By the Davis and Kahan sin θ Theorem, $\theta \leq \sin^{-1} (\| R[A, \tilde{W}_{pq}] \|_2 / g_{pq})$. It has been shown that $\hat{g}_{pq} \leq g_{pq}$. Therefore, inequality (19) must hold.

For infinitesimal perturbations, where $\Delta M \to 0$ and $\Delta K \to 0$, the residual norm $R[A, \tilde{W}_{pq}] \to 0$. The gap g_{pq}, associated with a subset of eigenvectors corresponding to a cluster of eigenvalues that are well separated from their complementary spectrum, is a finite positive number. Therefore, it follows from the sin θ theorem that $\theta \to 0$ as well. It is thus deduced that the subspace spanned by these eigenvectors is insensitive to perturbations, i.e., in the presence of infinitesimal perturbations, the subspace is changed in a continuous manner.

V. Example

Consider the five degree-of-freedom system of Fig. 5a and its modified configuration shown in Fig. 5b. Suppose that the lowest three natural frequencies of the unmodified system and their associated mode-shapes were found by modal tests. Assume that the mass matrix M and the incremental matrices ΔM and ΔK are given, whereas the stiffness matrix K is unknown. The problem of bounding the eigenvalues of this system from the given data has been investigated in Ref. 8. It was found that $\tilde{\lambda}_1$ and $\tilde{\lambda}_2$ must lie in the intervals [1644.06, 1707.60] and [11192.35, 14033.91], respectively. Thus, $\lambda_1^U = 1707.60$ and $\lambda_2^L = 11192.35$. It follows from Eqs. (5) and (7) that $\tilde{\lambda}_1 = 1707.60$, $F_{1,1} = (-0.99646, 0.08358, 0.00890)^T$, and $\tilde{W}_{1,1} = (0.59185, 0.32118, 0.48015, 0.26267, 0.49699)^T$. Therefore, by Eqs. (11) and (12) we have $\hat{g}_{1,1} = 9484.75$, and from the proposition of Sec. IV we obtain the residual vector $R[A, \tilde{W}_{1,1}] = (9.12, 456.19, -543.87, -454.02, 459.72)^T$. Hence, $\|R[A, \tilde{W}_{1,1}]\|_2 = 959.93$, and applying the main theorem, the desired bound is found to be $\theta \leq \sin^{-1} (959.93/9484.75) = 5.8$ deg.



a) Original system

b) Modified system

Fig. 5 Five degree-of-freedom systems.

To confirm the result, the (exact) eigenvector of the modified system corresponding to $\tilde{\lambda}_1$ was found (using the "missing data") to be $W_{1,1} = (0.57918, 0.30162, 0.50914, 0.28364, 0.48361)^T$. The angle between $W_{1,1}$ and its Rayleigh-Ritz approximation is, therefore,

$$\underline{/W_{1,1}, \tilde{W}_{1,1}} = \cos^{-1} \frac{| W_{1,1}^T \tilde{W}_{1,1} |}{\| W_{1,1} \|_2 \, \| \tilde{W}_{1,1} \|_2} = 2.56 < \theta$$

The physical interpretation of the result is that the approximated eigenvector lies in a (five-dimensional) conical section of an apex angle 5.8 deg whose axis of symmetry is the Rayleigh-Ritz vector $\tilde{W}_{1,1}$. Thus, a small angle θ guarantees the tightness of the approximation.

By using this result we may also bound the maximal variation of a specific element in the exact eigenvectors. Suppose for example that we wish to find the possible variation of the first element of $W_{1,1}$. Then, assuming that all the approximation error is concentrated in this element we obtain $\tilde{W}_{1,1} = (0.59185 \pm \delta, 0.32118, 0.48015, 0.26267, 0.49699)^T$, where δ represents the approximation error. Then $(| W_{1,1}^T \tilde{W}_{1,1} | / \|W_{1,1}\|_2 \|\tilde{W}_{1,1}\|_2) \leq \cos(5.8 \text{ deg})$ yields $\delta = 0.136$. The first element of $W_{1,1}$ is, therefore, $0.59185 + 0.136$. Such a bound may be useful when analyzing the stress and strain distribution in the vibrating structure.

VI. Conclusion

A method for bounding the approximation error of the mode shapes corresponding to a modified structure has been presented. The approximated mode shapes and their error bounds are based on incomplete modal data for the unmodified structure, such as usually available from modal testing, together with the analytically determined incremental mass and stiffness matrices. When the natural frequency is well separated, the bound is given in terms of the angle between the exact eigenvector and its approximation. The eigenvectors corresponding to poorly separated natural frequencies are highly sensitive to perturbation. Consequently, the bound in this case is on the angle between the subspace spanned by certain eigenvectors and its approximation.

In general, the inconsistency between analytical models and experimental results may be explained by a number of error mechanisms, e.g., nonlinearity, discretization of distributed systems, and uncertainties as to the physical parameters and boundary conditions. Approximation of the modal parameters of the modified structure based on 1) modal testing results of the unmodified structure and 2) analytical model of the incremental mass and stiffness matrices reduces the effect of uncertainties in the physical parameters and boundary conditions. However, the unavailability of a complete set of modal parameters for the unmodified structure introduces additionally, the "error of truncation." It is the effect of the truncation on the approximated mode shapes that has been bounded here.

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